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MONDAY, JANUARY 28, 1861.

JAMES HENTHORN TODD, D. D., President, in the Chair.

READ a letter from George M. Miller, Esq., announcing a donation of some osseous remains and antiquities, found in the works of the Great Southern and Western Railroad, near the King's-Bridge Terminus.

Parke Neville, Esq., C. E., presented an ancient sword, spear-head, and some human remains, found under the street opposite to St. Bride's Church.

It was moved by the Rev. S. Haughton; seconded by the Rev. Joseph Carson, D. D., and

RESOLVED,—That it be recommended to the Council to lay a Report before the Academy with respect to the Cunningham Fund, including,—

1. A history of the fund from its origin to the present time;

2. A list of the recipients of the proceeds of that fund, and Cunningham Medallists; with a statement of the purposes for which they received such fund or medal: and that the Council be requested to lay before the Academy its views with respect to the best mode of appropriating the Cunningham Fund in future.

MONDAY, FEBRUARY 11, 1861.

JAMES HENTHORN TODD, D. D., President, in the Chair.

James Foulis Duncan, M. D., Wyville Thomson, LL. D., and David Walker, M. D., were elected members of the Academy.

The REV. ROBERT CARMICHAEL, F. T. C. D., read a paper—

ON THE GENERAL THEORY OF THE INTEGRATION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

A COMPLETE theory of the genesis and solution of non-linear partial differential equations of the second and higher orders, in two or more independent variables, has been long a desideratum; and any considerable addition to our knowledge in this department of science can only be expected from the combined labours of those who have devoted some attention to this branch of the Integral Calculus. The following paper is intended as a small contribution towards the formation of such a theory. The solutions of non-linear partial differential equations are sought in the form of what are denominated 'complete primitives,' exhibiting arbitrary constants, as contradistinguished from 'general primitives,' exhibiting arbitrary functions. It is allowed that solutions in the latter form are more general; but then they are theoretically unattainable, unless in certain cases which, the equations being linear, admit of symbolic treatment, or which are susceptible of reduction by methods not universally applicable. The importance of the subject, partly in

connexion with the Calculus of Variations, and partly with the higher branches of mathematical physics, cannot well be overrated. It is well known that in the more advanced departments of mathematical physics one of the principal obstacles to our progress arises from the difficulty of integrating the partial differential equations which represent the conditions of the problem investigated; that these integrals have been sought in form of general primitives, which are in most cases unattainable; and that, if such forms of solution are attainable, a fresh difficulty arises upon our seeking to determine the forms of the arbitrary functions introduced. It is impossible now to determine what may be the physical value of the integrals of such differential equations, when stated in the form of complete primitives; but, in the default of better, such solutions may afford us, at least, *some* information. The following remarks are offered in the hope of being thus beneficial:—

1. If we are given any equation, including two arbitrary constants α , β , and two independent variables x , y ,

$$f(x, y, z, \alpha, \beta) = 0, \quad (\text{I})$$

we may differentiate this equation with respect to the independent variables x and y , thus obtaining

$$\left(\frac{df}{dx}\right) = 0, \left(\frac{df}{dy}\right) = 0;$$

and, by eliminating α , β between the three equations stated, we get a partial differential equation, in general non-linear,

$$F\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = F(x, y, z, p, q) = 0,$$

of which (I) is said to be a *complete primitive*.

2. Again if, u being any function of known form in x , y , z , we were given an equation of the type

$$f, \{x, y, z, \phi(u)\} = 0, \quad (\text{I}')$$

ϕ being any arbitrary function, we might differentiate as before, and eliminate ϕ , ϕ' , thus again arriving at a partial differential equation

$$F, \left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = F, (x, y, z, p, q) = 0,$$

of which (I') is said to be the *general primitive*.

3. Similarly, if we are given any equation including three arbitrary constants α , β , γ , and three independent variables x , y , z ,

$$f(x, y, z, w, \alpha, \beta, \gamma) = 0, \quad (\text{II})$$

we may differentiate this equation with respect to the independent variables x , y , z , thus obtaining

$$\left(\frac{df}{dx}\right) = 0, \left(\frac{df}{dy}\right) = 0, \left(\frac{df}{dz}\right) = 0;$$

and, by eliminating α, β, γ between the four equations stated, we would get a partial differential equation in three independent variables, in general non-linear,

$$F\left(x, y, z, w, \frac{dw}{dx}, \frac{dw}{dy}, \frac{dw}{dz}\right) = 0,$$

of which (II) would be said to be a complete primitive.

If, instead of differentiating with respect to the independent variables, we had differentiated with respect to the arbitrary constants, thus getting

$$\frac{df}{d\alpha} = 0, \frac{df}{d\beta} = 0, \frac{df}{d\gamma} = 0,$$

and then eliminated the arbitrary constants between the given equation and these three, we would obtain a result of the form

$$F(x, y, z, w) = 0,$$

which will, in general, satisfy the partial differential equation, previously derived,

$$F\left(x, y, z, w, \frac{dw}{dx}, \frac{dw}{dy}, \frac{dw}{dz}\right) = 0,$$

and, as it exhibits no arbitrary constants, may be denominated its singular solution.

4. Again if, u and v being any functions of known form in x, y, z, w , we were given an equation of the type

$$f_{II}\{x, y, z, w, \phi(u, v)\} = 0, \quad (\text{II}')$$

ϕ being any arbitrary function, we may differentiate with respect to the independent variables x, y, z , and eliminate

$$\phi, \frac{d\phi}{du}, \frac{d\phi}{dv},$$

thus again obtaining a partial differential equation

$$F_{II}\left(x, y, z, w, \frac{dw}{dx}, \frac{dw}{dy}, \frac{dw}{dz}\right) = 0,$$

of which (II') may be said to be the general primitive.

5. Proceeding by analogy, being given any equation in two independent variables, but exhibiting *five* arbitrary constants,

$$f(x, y, z, \alpha, \beta, \gamma, \delta, \epsilon) = 0, \quad (\text{III})$$

we may differentiate this equation twice successively with respect to x

and y , and eliminate the arbitrary constants between the given equation and

$$\left(\frac{df}{dx}\right) = 0, \left(\frac{df}{dy}\right) = 0, \left(\frac{d^2f}{dx^2}\right) = 0, \left(\frac{d^2f}{dxdy}\right) = 0, \left(\frac{d^2f}{dy^2}\right) = 0.$$

Thus we should obtain a partial differential equation of the second order, in general non-linear, of the form (adopting the ordinary notation),

$$F(x, y, z, p, q, r, s, t) = 0, \quad (\text{IV})$$

of which (III) may be said to be a complete primitive.

6. It does not follow, however, (as is known), that, in general, from a given finite equation exhibiting *two arbitrary functions*, we can pass to a partial differential equation of the second order. For if, u and v being any two functions of known form in x, y, z , we were given any equation of the type

$$f(x, y, z, \phi u, \psi v) = 0,$$

differentiating this equation twice successively with respect to x and y we only get six equations, which, ordinarily, are *not* sufficient to enable us to eliminate *the six* quantities

$$\phi, \phi', \phi'', \psi, \psi', \psi''.$$

It is true indeed that, in the case of *linear* partial differential equations of the second order, *which admit of treatment by symbolic methods*, we do, in most cases, obtain the solutions, in the form of general primitives, by direct procedures, and exhibiting two arbitrary functions. But the fact, which has just been alluded to, would appear to show that in the case of *non-linear* partial differential equations of the second and higher orders, which do not admit in general of treatment by symbolic methods, the species of solution which we should seek to obtain, should be, not the general primitive exhibiting arbitrary functions, but the complete primitive exhibiting arbitrary constants; and the mode of integration which we should seek to perfect, should be that by which the complete primitives of such partial differential equations are sought, and not their general primitives.

7. Moreover, we may differentiate the equation

$$f(x, y, z, \alpha, \beta, \gamma, \delta, \epsilon) = 0$$

with respect to the arbitrary constants, and between the five equations thus found, namely,

$$\frac{df}{d\alpha} = 0, \frac{df}{d\beta} = 0, \frac{df}{d\gamma} = 0, \frac{df}{d\delta} = 0, \frac{df}{d\epsilon} = 0,$$

and the given equation, eliminate $\alpha, \beta, \gamma, \delta, \epsilon$, thus obtaining a resulting equation of the form

$$F_1(x, y, z) = 0. \quad (\text{V})$$

It may be interesting to inquire what relation this equation bears to the corresponding partial differential equation (IV); in other words, whether there is any such relation between the two, as would justify us in denominating (V), *in any case*, the singular solution of (IV): what analytical conditions would be requisite in order that the former equation should satisfy the latter: and what may be the geometrical significance of singular solutions of partial differential equations of the second order, if such singular solutions are admissible or conceivable.

As regards the analytical conditions specified, they may be investigated thus. Differentiating the equation

$$f(x, y, z, a, \beta, \gamma, \delta, \epsilon) = 0$$

with respect to all the variables, we get identically

$$\left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy + \frac{df}{da}da + \frac{df}{d\beta}d\beta + \frac{df}{d\gamma}d\gamma + \frac{df}{d\delta}d\delta + \frac{df}{d\epsilon}d\epsilon = 0,$$

which, in consequence of the relations supposed, reduces to

$$\left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy = 0,$$

or, as dx and dy are independent, we have the first two of the relations stated in the fifth article, namely,

$$\left(\frac{df}{dx}\right) = 0, \left(\frac{df}{dy}\right) = 0.$$

So far all is plain; but when we proceed to differentiate these equations again with respect to all the variables, we get

$$\begin{aligned} \left(\frac{d^2f}{dx^2}\right)dx + \left(\frac{d^2f}{dx dy}\right)dy + \left(da \frac{d}{da} + d\beta \frac{d}{d\beta} + d\gamma \frac{d}{d\gamma} + d\delta \frac{d}{d\delta} + d\epsilon \frac{d}{d\epsilon}\right)\left(\frac{df}{dx}\right) &= 0 \\ \left(\frac{d^2f}{dy^2}\right)dy + \left(\frac{d^2f}{dx dy}\right)dx + \left(da \frac{d}{da} + d\beta \frac{d}{d\beta} + d\gamma \frac{d}{d\gamma} + d\delta \frac{d}{d\delta} + d\epsilon \frac{d}{d\epsilon}\right)\left(\frac{df}{dy}\right) &= 0, \end{aligned}$$

which are not equivalent to the remaining relations of the fifth article, namely,

$$\left(\frac{d^2f}{dx^2}\right) = 0, \left(\frac{d^2f}{dx dy}\right) = 0, \left(\frac{d^2f}{dy^2}\right) = 0,$$

unless, simultaneously,

$$\left. \begin{aligned} \left(da \frac{d}{da} + d\beta \frac{d}{d\beta} + d\gamma \frac{d}{d\gamma} + d\delta \frac{d}{d\delta} + d\epsilon \frac{d}{d\epsilon}\right) \cdot \left(\frac{df}{dx}\right) &= 0 \\ \left(da \frac{d}{da} + d\beta \frac{d}{d\beta} + d\gamma \frac{d}{d\gamma} + d\delta \frac{d}{d\delta} + d\epsilon \frac{d}{d\epsilon}\right) \cdot \left(\frac{df}{dy}\right) &= 0 \end{aligned} \right\}$$

8. It is known that, if we are given any system of surfaces explicable by the partial differential equation of the first order

$$F(x, y, z, p, q) = 0,$$

we can, in general, determine the character of this system by finding the integral of the differential equation, either in the form of a complete primitive, or a general primitive.

Again, if we are given any system of surfaces explicable by the partial differential equation of the second order

$$F(x, y, z, p, q, r, s, t) = 0,$$

we may, in general, determine the character of this system by finding, if we can, the integral of the differential equation, either in the form of a complete primitive, or a general primitive.

If the solution, in either of these two cases, be obtained in the form of a general primitive, the form of the arbitrary function or functions introduced is, in general, determined by supposing the surface to pass through a given curve or curves. The difficulty of applying this principle, in practice, is well known.

If, on the other hand, the solution, in either case, be obtained in the form of a complete primitive, everything required by the solution is determined if, in the former case, *two points* upon the surface represented by the partial differential equation be given, and, in the latter case, *five points*. More generally, if the given non-linear partial differential equation be of the first order and include n independent variables, it is sufficient, for the determination of the function represented, in the form of a complete primitive, that we be given n systems of correspondent values of the variables. If the given non-linear partial differential equation be of the second order, and include n independent variables, it is sufficient for the complete determination of the function represented, that we be given $\frac{n(n+3)}{2}$ systems of correspondent values of the variables.

9. I now proceed to discuss certain examples with the view of showing the *feasibility* of obtaining solutions, in the form of complete primitives, of non-linear partial differential equations of the second order, and, in the following article, a general method for deriving such solutions will be indicated: the completion of the subject is reserved for a supplementary communication.

Examples.

(1). Let it be proposed to determine the solution, in the form of a complete primitive, of the non-linear partial differential equation

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (1)$$

The auxiliary system, adopting the method of Lagrange and Charpit, gives

$$dp = 0,$$

since

$$\frac{dq}{dx} = 0, \quad \frac{dq}{dz} = 0.$$

Hence, c being any arbitrary constant,

$$p = c, \quad \text{and} \quad q = \frac{c}{c-1};$$

and substituting these values in the general equation

$$dz = p dx + q dy$$

we get, writing $\frac{1}{a}$ for c ,

$$z = \frac{x}{a} + \frac{y}{1-a} + \beta. \quad (2)$$

This then is a complete primitive of the given equation, and it is at once seen to satisfy it. But, moreover, if we examine the other equations of the auxiliary system in this case, namely,

$$(p-1)^2 dx = dy = \left(\frac{p-1}{p}\right)^2 dz,$$

we see that no other complete primitive is deducible. In other words, equation (2) is *the* complete primitive of (1).

If β be made an arbitrary function of a , we get as the general primary of (1)

$$\left. \begin{aligned} z &= \frac{x}{a} + \frac{y}{1-a} + \phi(a) \\ 0 &= \frac{x}{a^2} - \frac{y}{(1-a)^2} - \phi'(a) \end{aligned} \right\}$$

(2). If it be now proposed to integrate the higher differential equa-

tion in three independent variables (and a corresponding method will apply generally)

$$\frac{1}{\frac{dw}{dx}} + \frac{1}{\frac{dw}{dy}} + \frac{1}{\frac{dw}{dz}} = 1,$$

let us assume for w the linear form

$$w = \frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma} + \kappa,$$

where a, β, γ, κ are arbitrary constants. Then, by substitution, we find that this will be the solution required, if the arbitrary constants be connected by the relation

$$a + \beta + \gamma = 1.$$

Hence the complete primitive of the given equation is, writing γ for κ ,

$$w = \frac{x}{a} + \frac{y}{\beta} + \frac{z}{1 - (a + \beta)} + \gamma,$$

and the general primary

$$\left. \begin{aligned} w &= \frac{x}{a} + \frac{y}{\beta} + \frac{z}{1 - (a + \beta)} + \phi(a, \beta) \\ 0 &= \frac{d}{da} \left\{ \frac{x}{a} + \frac{y}{\beta} + \frac{z}{1 - (a + \beta)} + \phi(a, \beta) \right\} \\ 0 &= \frac{d}{d\beta} \left\{ \frac{x}{a} + \frac{y}{\beta} + \frac{z}{1 - (a + \beta)} + \phi(a, \beta) \right\} \end{aligned} \right\}$$

I have ventured to denominate the system of equations just written, the 'general primary' solution of the given partial differential equation, instead of, as it is usually denominated, the 'general primitive.' It would seem to be right to distinguish between the two cases, where one arbitrary constant is made an arbitrary function of one or more others, and where an arbitrary function of the variables is introduced. It is true, indeed, that in some cases these duplicate solutions coincide, as, for instance, in the case of the general functional equation of surfaces of revolution, namely,

$$lx + my + nz = \phi(x^2 + y^2 + z^2)$$

which may be readily identified with the general primary, obtained from the complete primitive

$$(x - c \cos \lambda)^2 + (y - c \cos \mu)^2 + (z - c \cos \nu)^2 = \gamma^2,$$

by putting $c = \psi(r)$; but such identification does not appear to be generally possible.

It is easy to see how the following non-linear partial differential equations are but modifications of the simple examples treated, namely,

$$\left. \begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{k}{z} \\ \frac{x}{p} + \frac{y}{q} &= kz \\ \frac{x^m}{p} + \frac{y^m}{q} &= kz^m \end{aligned} \right\} ; \text{ or } \left\{ \begin{aligned} \frac{1}{\frac{dw}{dx}} + \frac{1}{\frac{dw}{dy}} + \frac{1}{\frac{dw}{dz}} &= \frac{k}{w} \\ \frac{x}{\frac{dw}{dx}} + \frac{y}{\frac{dw}{dy}} + \frac{z}{\frac{dw}{dz}} &= kw, \\ \frac{x^m}{\frac{dw}{dx}} + \frac{y^m}{\frac{dw}{dy}} + \frac{z^m}{\frac{dw}{dz}} &= kw^m, \end{aligned} \right.$$

the solutions being, respectively,

$$\left. \begin{aligned} \log z &= \frac{1}{k} \left(\frac{x}{a} + \frac{y}{1-a} + \beta \right) \\ z^2 &= \frac{1}{k} \left(\frac{x^2}{a} + \frac{y^2}{1-a} + \beta \right) \\ z^{m+1} &= \frac{1}{k} \left(\frac{x^{m+1}}{a} + \frac{y^{m+1}}{1-a} + \beta \right) \end{aligned} \right\}$$

and

$$\left. \begin{aligned} \log w &= \frac{1}{k} \left(\frac{x}{a} + \frac{y}{\beta} + \frac{z}{1-(a+\beta)} + \gamma \right) \\ w^2 &= \frac{1}{k} \left(\frac{x^2}{a} + \frac{y^2}{\beta} + \frac{z^2}{1-(a+\beta)} + \gamma \right) \\ w^{m+1} &= \frac{1}{k} \left(\frac{x^{m+1}}{a} + \frac{y^{m+1}}{\beta} + \frac{z^{m+1}}{1-(a+\beta)} + \gamma \right) \end{aligned} \right\}$$

(3). If it were proposed to integrate the non-linear partial differential equation of the second order, in two independent variables,

$$\frac{1}{r} + \frac{1}{t} = \frac{1}{\frac{d^2z}{dx^2}} + \frac{1}{\frac{d^2z}{dy^2}} = 1,$$

we would assume, by analogy, the solution to be

$$z = \frac{x^2}{2a} + \frac{y^2}{2(1-a)} + \frac{xy}{\beta} + \frac{x}{\gamma} + \frac{y}{\delta} + \epsilon;$$

and it is instantly evident that this is a complete primitive of the given equation, since it satisfies it, and exhibits five arbitrary constants.

Similarly, the solution, in the form of a complete primitive, of the non-linear partial differential equation of the second order, in three independent variables,

$$\frac{1}{\frac{d^2w}{dx^2}} + \frac{1}{\frac{d^2w}{dy^2}} + \frac{1}{\frac{d^2w}{dz^2}} = 1,$$

is

$$w = \frac{x^2}{2a} + \frac{y^2}{2\beta} + \frac{z^2}{2(1-a+\beta)} + \frac{yz}{\gamma} + \frac{zx}{\delta} + \frac{xy}{\epsilon} + \frac{x}{\xi} + \frac{y}{\eta} + \frac{z}{\theta} + \kappa,$$

since, as before, it satisfies the given partial differential equation, and exhibits nine arbitrary constants.

It is readily seen that if the former of the two partial differential equations now treated, had been

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{1}{\frac{d^2z}{dx^2}} + \frac{1}{\frac{d^2z}{dx dy}} + \frac{1}{\frac{d^2z}{dy^2}} = 1,$$

we should have, for a complete primitive, the obvious modification

$$z = \frac{x^2}{2a} + \frac{y^2}{2\beta} + \frac{xy}{1-(a+\beta)} + \frac{x}{\gamma} + \frac{y}{\delta} + \epsilon.$$

(4). It follows readily from the forms of solution arrived at in the examples just discussed, that the complete primitive of any non-linear partial differential equation of the type

$$f(p, q) = 0,$$

is the linear expression

$$z = ax + by + c,$$

where a and b are connected by the equation

$$f(a, b) = 0.$$

Thus the complete primitive of the partial differential equation

$$\frac{1}{p^m} + \frac{1}{q^m} = k$$

is

$$z = \frac{x}{a^{\frac{1}{m}}} + \frac{y}{(1-a)^{\frac{1}{m}}} + \beta.$$

More generally, the complete primitive of the partial differential equation in three independent variables

$$\frac{1}{\left(\frac{dw}{dx}\right)^m} + \frac{1}{\left(\frac{dw}{dy}\right)^m} + \frac{1}{\left(\frac{dw}{dz}\right)^m} = 1$$

is

$$w = \frac{x}{a^{\frac{1}{m}}} + \frac{y}{\beta^{\frac{1}{m}}} + \frac{z}{(1-a+\beta)^{\frac{1}{m}}} + \gamma.$$

Similarly, the integral of the partial differential equation

$$\frac{1}{r^m} + \frac{1}{t^m} = \frac{1}{\left(\frac{d^2z}{dx^2}\right)^m} + \frac{1}{\left(\frac{d^2z}{dy^2}\right)^m} = 1,$$

in the form of a complete primitive, is

$$z = \frac{x^2}{2a^{\frac{1}{m}}} + \frac{y^2}{2(1-a)^{\frac{1}{m}}} + \frac{xy}{\beta} + \frac{x}{\gamma} + \frac{y}{\delta} + \epsilon;$$

and the solution of

$$\frac{1}{\left(\frac{d^3w}{dx^3}\right)^m} + \frac{1}{\left(\frac{d^3w}{dy^3}\right)^m} + \frac{1}{\left(\frac{d^3w}{dz^3}\right)^m} = 1$$

is

$$w = \frac{x^3}{2a^{\frac{1}{m}}} + \frac{y^3}{2\beta^{\frac{1}{m}}} + \frac{z^3}{2(1-a+\beta)^{\frac{1}{m}}} + \frac{yz}{\gamma} + \frac{zx}{\delta} + \frac{xy}{\epsilon} + \frac{x}{\zeta} + \frac{y}{\eta} + \frac{z}{\theta} + \kappa.$$

So, again, the complete primitive of the partial differential equation

$$\frac{1}{r^m} + \frac{1}{s^m} + \frac{1}{t^m} = 1,$$

is

$$z = \frac{x^2}{2a^{\frac{1}{m}}} + \frac{y^2}{2\beta^{\frac{1}{m}}} + \frac{xy}{(1-a+\beta)^{\frac{1}{m}}} + \frac{x}{\gamma} + \frac{y}{\delta} + \epsilon.$$

(5.) It may be interesting to compare for a particular case the solution in the form of a complete primitive with the solution in the form of a general primitive. Thus let us take the equation

$$r + 2s + t = \frac{d^2z}{dx^2} + 2 \frac{d^2z}{dxdy} + \frac{d^2z}{dy^2} = 1.$$

The solution of this equation by the symbolic method, in the form of a general primitive, is readily found to be

$$z = \frac{xy}{2} + \frac{x+y}{2} \cdot \phi(x-y) + \psi(x-y).$$

The solution of the same equation, in the form of a complete primitive, is

$$z = \frac{1}{2} \{ ax^2 + \beta y^2 + (1 - a + \beta) xy \} + \gamma x + \delta y + \epsilon.$$

These solutions may be readily identified by assuming

$$\begin{aligned} \phi(x-y) &= A(x-y) + A', \\ \psi(x-y) &= B(x-y)^2 + B'(x-y) + B'', \end{aligned}$$

whence, by substitution in the former, we get

$$z = \frac{xy}{2} + \frac{1}{2} \{ A(x^2 - y^2) + A'(x+y) \} + \{ B(x-y)^2 + B'(x-y) + B'' \}$$

and, by comparison of co-efficients,

$$\begin{aligned} \frac{1}{2}A + B &= \frac{1}{2}\alpha, \quad B - \frac{1}{2}A = \frac{1}{2}\beta, \quad 2B = \frac{1}{2}(\alpha + \beta), \\ \frac{1}{2}A' + B' &= \gamma, \quad \frac{1}{2}A' - B' = \delta, \quad B'' = \epsilon. \end{aligned}$$

A similar identification may be performed upon the solutions, in the two forms, of the equation

$$r + t = 1.$$

Thus it appears that the general primitive may be reduced to the complete primitive, but not *vice versa*.

(6.) The following will serve as examples of the latter portion of the third article. Let the partial differential equation proposed for solution be

$$x \frac{dw}{dx} + y \frac{dw}{dy} + z \frac{dw}{dz} = \left\{ a^m \left(\frac{dw}{dx} \right)^m + b^m \left(\frac{dw}{dy} \right)^m + c^m \left(\frac{dw}{dz} \right)^m \right\}^{\frac{1}{m}},$$

where a, b, c are given constants: then the solution of this equation, in the form of a complete primitive, is

$$w = \alpha x + \beta y + \gamma z - \{ a^m \alpha^m + b^m \beta^m + c^m \gamma^m \}^{\frac{1}{m}} = 0,$$

and the singular solution is

$$\left(\frac{x}{a} \right)^{\frac{m}{m-1}} + \left(\frac{y}{b} \right)^{\frac{m}{m-1}} + \left(\frac{z}{c} \right)^{\frac{m}{m-1}} = 1.$$

Thus, if $m = 3$, the singular solution of the partial differential equation

$$x \frac{dw}{dx} + y \frac{dw}{dy} + z \frac{dw}{dz} = \left\{ a^3 \left(\frac{dw}{dx} \right)^3 + b^3 \left(\frac{dw}{dy} \right)^3 + c^3 \left(\frac{dw}{dz} \right)^3 \right\}^{\frac{1}{3}}$$

is

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1.$$

Again, if $m = \frac{1}{3}$, the singular solution of

$$x \frac{dw}{dx} + y \frac{dw}{dy} + z \frac{dw}{dz} = \left\{ \left(a \frac{dw}{dx} \right)^{\frac{1}{3}} + \left(b \frac{dw}{dy} \right)^{\frac{1}{3}} + \left(c \frac{dw}{dz} \right)^{\frac{1}{3}} \right\}^3$$

is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Finally, it is easily proved that the singular solution of the partial differential equation

$$a \frac{dw}{dx} \left\{ x \frac{dw}{dx} + y \frac{dw}{dy} + z \frac{dw}{dz} \right\} + b^2 \left(\frac{dw}{dy} \right)^2 + c^2 \left(\frac{dw}{dz} \right)^2 = 0$$

is the paraboloid

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{4x}{a}.$$

Generalisation of Charpit's Method.

10. Let us suppose that the given non-linear partial differential equation is reduced to the form

$$s = f(x, y, z, p, q, r, t).$$

Then we have the two conditions

$$\left(\frac{dr}{dy} \right) = \left(\frac{ds}{dx} \right), \quad \left(\frac{dt}{dx} \right) = \left(\frac{ds}{dy} \right).$$

Now on supposition that r is expressed generally as a function of x, y, z, p , and q , we have

$$\left(\frac{dr}{dy} \right) = \frac{dr}{dy} + \frac{dr}{dz} q + \frac{dr}{dp} s + \frac{dr}{dq} t,$$

and

$$\begin{aligned} \left(\frac{ds}{dx} \right) = & \frac{ds}{dx} + \frac{ds}{dz} p + \frac{ds}{dp} r + \frac{ds}{dq} s + \frac{ds}{dr} \left(\frac{dr}{dx} + \frac{dr}{dz} p + \frac{dr}{dp} r + \frac{dr}{dq} s \right) \\ & + \frac{ds}{dt} \left(\frac{dt}{dx} + \frac{dt}{dz} p + \frac{dt}{dp} r + \frac{dt}{dq} s \right). \end{aligned}$$

Hence, equating and arranging terms, we get a linear partial differential equation in r and t as dependent variables, and x, y, z, p, q as independent variables, namely,

$$\left. \begin{aligned} & -\frac{ds}{dr} \cdot \frac{dr}{dx} + \frac{dr}{dy} + \left(q - p \frac{ds}{dr} \right) \frac{dr}{dz} + \left(s - r \frac{ds}{dr} \right) \frac{dr}{dp} + \left(t - s \frac{ds}{dr} \right) \frac{dr}{dq} \\ & - \frac{ds}{dt} \left(\frac{dt}{dx} + p \frac{dt}{dz} + r \frac{dt}{dp} + s \frac{dt}{dq} \right) = \frac{ds}{dx} + \frac{ds}{dz} p + \frac{ds}{dp} r + \frac{ds}{dq} s \end{aligned} \right\} (\alpha)$$

in which, of course, the co-efficients $\frac{ds}{dx}, \frac{ds}{dy}, \frac{ds}{dz}, \frac{ds}{dp},$ &c., are all known functions of $x, y, z, p, q, r,$ and t .

The second condition gives the corresponding equation

$$\left. \begin{aligned} & \frac{dt}{dx} - \frac{ds}{dt} \cdot \frac{dt}{dy} + \left(p - q \frac{ds}{dt} \right) \frac{dt}{dz} + \left(r - s \frac{ds}{dt} \right) \frac{dt}{dp} + \left(s - t \frac{ds}{dt} \right) \frac{dt}{dq} \\ & - \frac{ds}{dr} \left(\frac{dr}{dy} + q \frac{dr}{dz} + s \frac{dr}{dp} + t \frac{dr}{dq} \right) = \frac{ds}{dy} + \frac{ds}{dz} q + \frac{ds}{dp} s + \frac{ds}{dq} t \end{aligned} \right\} (\beta)$$

in which, as before, the co-efficients $\frac{ds}{dy}, \frac{ds}{dz},$ &c., are known functions of $x, y, z, p, q, r,$ and t .

This system of simultaneous partial differential equations may be much simplified by writing

$$\begin{aligned} \frac{d}{dx} + p \frac{d}{dz} + r \frac{d}{dp} + s \frac{d}{dq} &= X, \\ \frac{d}{dy} + q \frac{d}{dz} + s \frac{d}{dp} + t \frac{d}{dq} &= Y, \\ \frac{ds}{dx} + p \frac{ds}{dz} + r \frac{ds}{dp} + s \frac{ds}{dq} &= f_1(x, y, z, p, q, r, t) \\ \frac{ds}{dy} + q \frac{ds}{dz} + s \frac{ds}{dp} + t \frac{ds}{dq} &= f_2(x, y, z, p, q, r, t) \\ \frac{ds}{dr} = R, \quad \frac{ds}{dt} &= T. \end{aligned}$$

By making these substitutions, the given system of simultaneous partial differential equations becomes, simply,

$$\left. \begin{aligned} Y \cdot r - RX \cdot r - TX \cdot t &= f_1 \\ X \cdot t - TY \cdot t - RY \cdot r &= f_2 \end{aligned} \right\}$$

Hence, in general, we have a system of two simultaneous linear partial differential equations to determine r and t in terms of p, q, x, y, z . Supposing these found, it remains to substitute their values in

$$\left. \begin{aligned} dp &= rdx + sdy \\ dq &= sdx + tdy \end{aligned} \right\}$$

and integrate again. The values of p and q thus found are to be substituted, finally, in the equation

$$dz = pdx + qdy,$$

and by a third integration the solution required is, in general, determined. It will be evident that in the processes of successive integration indicated, five arbitrary constants are introduced.

JOSEPH HUBAND SMITH, Esq., read a paper—

ON THE ANCIENT NORSE AND DANISH GEOGRAPHY OF IRELAND.

AFTER some observations upon communications made to the Academy by the late Mr. George Downes, in the years 1838-9 and 1841, and having referred also to the labours of Dr. William West, who had died in 1837, while engaged in preparing an Essay on the Ancient Geography of Gaul and the British Isles, intended to be laid before the meeting of the British Association, then approaching, Mr. Smith proceeded to read some passages from recently-discovered copies of our ancient annals, illustrative of the military occupation of Ireland by the Vikings, who arrived from Scandinavia at various periods, commencing towards the close of the eighth century of our era, and continued a long series of contests with the aboriginal inhabitants of Ireland, and subsequently with each other, during the four succeeding centuries.

From passages in the Icelandic and other Sagas, he showed that a constant intercourse had been maintained at first by the Norwegian sea rovers, who had gained a footing in the Orkneys, and afterwards by others, described in the Irish annals by the names of Lochlanns, Gentiles, and Finn Ghaoill, and Dubh Ghaoill, not only of a warlike nature, but also of a trading or mercantile character. The names of various localities which he instanced,—beginning with the Orkneys, the Faro-Islands, and the Hebrides,—all, he contended, indicated the progress of the war vessels of the various tribes; and he pointed out the harbours and islands along the eastern coast of Ireland, whose names (some as recorded in the Sagas, and others preserved to the present day) were unmistakeably of Norwegian and Icelandic origin, and quite distinct from the Celtic names by which these places were known to the Irish writers of the same period. Among others, Mr. Smith instanced Strangford bay, Carlingford, the little islands of Lamb-ey, and Ireland's-ey; the harbour of Bullock or Blowyk, and the adjacent island and sound of Dalkey; the headlands of Wicklow, or Wykynge-lo, and Arc-lo; the harbours of Wexford and Waterford, as well as that of Smerwick, on the western coast, not far from the River Shannon,—in all of which the Norwegian or Danish Vikings had sheltered successive fleets, and in most cases erected on their shores military fortresses of considerable strength. Some other places named in the Sagas,—for example, Gunvallsborg,—Mr. Smith admitted he was as yet unable to identify satisfactorily with any modern localities. He then adverted to the modern names of three of